Sets versus Classes

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The modern iterative concept of set has, as Charles Parsons has remarked, no direct relation to the notions of predicate or definable property. Sets are thought of as being constructed by a transfinite process which begins with the empty set. At each stage of this process all sets are formed whose members have already been formed at earlier stages. The length of the transfinite process is absolute infinity. The class of all sets is not a completed totality: it is absolutely potential.

The lack of a direct connection between sets and predicates helps in two ways to avoid possible problems about sets. In one direction, the paradoxes do not arise. Non-self-membership is a definable property of sets, but this does not mean that there is a set of all sets which are not members of themselves. Such a set would have to be constructed at some stage in the transfinite process, and it is easy to see that this cannot happen. In the other direction, much of the traditional concern about the axiom of choice is probably based on a confusion between sets and definable properties. In many cases it appears unlikely that one can define a choice function for a particular collection of sets. But this is entirely unrelated to the question of whether a choice function exists. Once this kind of confusion is avoided, the axiom of choice appears as one of the least problematic of the set theoretic axioms.

Nevertheless, as soon as one starts doing set theory -- in particular, formal axiomatic set theory -- problems about predicates, properties, and proper classes begin to creep back in.

The first problem arising is that of quantifying over the class of all sets. I am going to try to argue that this is not a real problem, but I wish to say in advance that I am not entirely convinced that I am right.

One argument against quantifying over the class of all set runs as follows: If one is to quantify over a domain, the domain over which one quantifies must be a completed totality, [i. e., a set.] Just as intuitionists hold that the natural numbers are not a completed totality and therefore reject classical quantification over the natural numbers, so set theorists should reject classical quantification over the class of all sets. ?

I confess that I do not see the force of this argument. I don't see why one must believe in infinite sets to understand an assertion like $(\forall x)(\exists y)(\forall z)(\exists w) R(x, y, z, w)$ where R is, say, a molecular formula in the usual formal number theory. The assertion -- and the quantifiers -- do not directly mention any infinite totality. I do not see that

* I agree - if we interpret this statement about e.g. possible Turing machine. But if we think it is a statement about objects it is hard to avoid this.

* The need arises at the metalevel - in the semantic analysis of the claim

X It seems to me that the only [illegible] a convincing semantics which expresses this idea. This seems also the view held by Gödel – cf. BP, 262

the assertion presupposes the existence of any infinite totality. It presupposes only certain finite objects, i.e. each of the natural numbers. Similarly, I do not see how the intelligibility of any particular assertion in the language of formal set theory depends upon the assumption of a completed totality, the class of all sets. I should remark, first of all, that I am merely reporting my intuitions about this matter and, secondly, that I am making no claims about the intelligibility of <u>schemata</u> of formal set theory. X

Another argument against quantification over the class of all sets has been given by Charles Parsons. Suppose I explain what I mean by set and announce that I believe in the Zermelo-Fraenkel axioms. You then introduce to me the notion of an inaccessible cardinal and give arguments so that I am led to add to my list of axioms for set theory the assertion that inaccessible cardinals exist. Can you not maintain that I have changed my concept of set -- changed my universe of discourse -- and that my old universe of discourse was the set of all sets of rank less than the least inaccessible cardinal? This would indeed be fair if the only thing I had managed to express to you about my notion of set was that the Z-F axioms were true of it. But hopefully I was able to convey more than this about the concept. Surely I was able to make clear that -- whatever else was true about sets -- the range of the variables of my set theory was not a completed totality -- could not be a set in some more inclusive concept of set. Though in some sense my coming to hold that inaccessibles exist has perhaps changed my concept of set, the notion of all-inclusiveness, of absolute infinity, was already a central part of my concept.

Whether or not I am right that problems about proper classes are not involved in particular set-theoretic assertions, as soon as schemata or proper classes as a formal device are introduced, problems do arise. When a set theorist introduces the axiom schema of replacement or lists some schema as a "theorem" he will usually give the "safe" explanation that he wishes to convey only that each of the instances of the schema are axioms or theorems of his formal system. But, as Parsons remarks, what the set theorist really wishes to do -- and what he seemingly succeeds in doing -- is stimultaneously to assert each of the propositions which the instances of the schema express. The only way he can accomplish this simultaneous assertion seems to be to assert that each of the instances of the schema is true. Similarly, when a set theorist writes down theorems of set theory involving proper classes, he may explain that by "proper class" he means only "first order definable proper class" and that his intention is merely to indicate in an elegant way that certain sentences are theorems of Z-F set theory. But in fact his intention is surely to do more: either to make direct assertions about proper classes themselves or else to indicate that each of a certain infinite collection of sentences of Z-F is true.

In doing set theory, we are thus led in the most natural way to an involvement with the truth predicate for set theory and with the satisfaction predicate for set theory. Such involvement appears to destroy the security which we gained for ourselves in avoiding the paradoxes via the iteration concept of set.

If we talk of the truth of a particular set theoretic sentence ϕ we can explain ourselves by saying that "ϕ is true" means the same thing as ϕ. There are even devices for allowing us to talk of truth for the collection of all sentences containing no more than some fixed number of quantifiers. But if we wish to talk of truth or satisfaction for, say, a schema, then we are led -- as is well known -- beyond the bounds of formal ZF set theory.

There are various ways to define truth in a particular structure, for example the standard model of number theory. We can say that certain Skolem functions exist. $(\forall x)(\exists y)(\forall z)(\exists w) R(x, y, z, w)$ is true just in case there are functions $f(x)$ and $g(x, z)$ such that $(\forall x)(\forall z) R(x, f(x), z, g(x, z))$. Alternatively, we might talk of a tree which witnesses the truth of the sentence in question:

$$(\forall x)(\exists y)(\forall z)(\exists w)\,R(x, y, z, w)$$

$$(\exists y)(\forall z)(\exists w)\,R(0, y, z, w) \qquad (\exists y)(\forall z)(\exists w)\,R(1, y, z, w) \qquad \ldots$$

$$(\forall z)(\exists w)\,R(0, y_0, z, w) \qquad (\forall z)(\exists w)\,R(1, y_1, z, w)$$

$$\ldots \qquad \ldots$$

$$(\exists w)\,R(0, y_0, 0, w)$$

$$R(0, y_0, 0, w_{00})$$

The tree witnesses the truth of our sentence just in case all the terminal nodes are true, i. e., are satisfied by the particular numbers which have replaced the free variables. A third way to express truth is to talk of a satisfaction predicate meeting certain inductive conditions. By each method, to say that a number theoretic assertion is true is to say that a certain infinite object exists: a Skolem function, a tree, a satisfaction predicate. To talk of the truth of number theoretic sentences seems then to involve talk about infinite sets. In the case of number theory, this is no problem, since we can talk about truth for number theory in analysis or set theory. In the case of set theory itself, there is a significant problem. The Skolem function, the truth tree, or the satisfaction predicate for a set-theoretic assertion would have to be a proper class, the very thing the iterative concept of set was supposed to avoid. On the one hand, it is easy to become convinced that there is nothing wrong with talking of a satisfaction predicate for set theory. On the other hand, it is difficult to explain why we are so easily convinced there is nothing wrong.

In these days of stronger and stronger large cardinal axioms, it has become common to talk not only of such garden variety proper classes as the satisfaction predicate for set theory but also of proper classes of higher rank. One thinks of the class of all sets as the rank R(On) and then speaks of its power set, R(On + 1). The whole iteration procedure is continued beyond On and one has R(On + α) for ordinals α. Sometimes even R(On + α) for α an ordinal beyond On is discussed. How does one make sense of such talk?

One solution is by a relativism of the kind espoused by Parsons. We think of our universe of discourse as ambiguous, so that when we speak of R(On + α) we are taking On to be an ordinary ordinal, though perhaps an ordinal larger than any hitherto envisaged. As I have indicated, I am not too happy with this solution. I would rather try to hang on to the absoluteness of absolute infinity. The construction of R(On) is absolutely incapable of completion. One cannot even begin the construction of R(On + 1). If there is to be a justification of proper classes, it must -- according to my absolutist view -- come

Parsons' [illegible] with this.

X This is [illegible] view which takes classes to be "big sets"

XX Parsons's view does not seem incompatible with an absolutist view – on Parsons' view all there is [illegible] the way of "classes" [illegible] in the cumulative hierarchy; there need be no actual R(On) – we simply [illegible] as if it were – this is OK for a given concept of set (e.g. [illegible]) from some direction other than the metaphor of construction.

Another solution is to explain proper classes as useful fictions. When we speak of $R(On + \alpha)$ we do not wish to imply that such a thing exists. We talk of such fictional entities only to motivate such things large cardinal axioms. For example, by reflection principles there should be larger and larger stages, ranks R_α, which look more and more like the universe of all sets $R(On)$. This similarity should extend to second and higher order properties, so that there should be arbitrarily large α such that $R(\alpha + 1)$ is an elementary substructure of $R(On + 1)$. It follows that there are ordinals α and β such that $R(\alpha + 1)$ is an elementary substructure of $R(\beta + 1)$. But this last assertion is an assertion about <u>sets</u>. Having arrived at it, we can throw away the prop, $R(On + 1)$, which was used as motivation. Now I am not opposed to such a method for discovering large cardinal axioms, and the method does seem to give the axioms discovered a certain amount of plausibility. The procedure seems in many ways like the use of infinitesimals to develop the calculus. It is quite fruitful and perhaps necessary, but its justification is a problem for the future.

My suggestion -- and it is only a suggestion, since I have at present no developed theory -- is not to try to justify classes in terms of sets at all but to develop the theory of classes directly. As Parsons has argued, sets are not directly explicable in terms of classes, so it seems to me that classes are not directly explicable in terms of sets. Sets are generated by an iterative construction processes. Classes are given all at once, by the properties that determine which objects are members of them.

To develop a theory of classes, at least if we allow classes which are members of classes or if we wish to talk about truth in the theory, is to come directly to grips with the paradoxes, and that is what I propose should be done. The suggestions I have in this direction are, I should remark, not really new.

The basic ingredients of a theory of classes are these: We allow sets and classes. Sets are the usual sets; their members are sets only. Classes can have as members both sets and classes. We want to be able to define a truth or satisfaction predicate, applying to Gödel numbers of formulas together with finite sequences of sets and classes (corresponding to the free variables of the formulas). If we cannot define a truth predicate, we are left with the same problems in our class theory as we had in classless set theory. We allow extensionality axioms and an unrestricted comprehension axiom for classes. Obviously we must make some concession to avoid the paradoxes (because of the definability of truth and the unrestricted comprehension axiom). The concession I have in mind is that not all sentences will have a truth value.

To see why this concession is natural, let us consider the definition of truth. A sentence ϕ is true if there is a tree which witnesses its truth. This tree, construed as a collection of n-tuples of classes, is to be a class. By a tree witnessing the truth of ϕ I mean a tree which constitutes a proof of ϕ, according to certain rules of inference. For example, $(\forall X)(\phi(X))$ follows from the collection of formulas $\phi[A]$ for all classes A. Of course, the law of the excluded middle does not hold. A tree constitutes a proof of ϕ if (1) ϕ is the top node of the tree, (2) each node of the tree follows from the sentences immediately below it by one of the rules of inference, (3) terminal nodes are true atomic formulas or their negation, and (4) the tree is well-founded. Instances where well-foundedness is relevant are the sentence which says that the Russell class belongs to itself, whose truth follows from the truth of its negation, which in turn follows from its truth, etc., and the Gödel sentence, "I am not true." Even the sentence "ϕ has a truth value" may have no definite truth value.

A theory of classes built along these lines will be fruitful if we can sharpen our concept of class or property sufficiently so that a strong set of axioms is produced and so that we feel we are dealing with something fairly definite. For example, can we come to a clear-cut decision as to the existence of $R(On + 1)$ and $R(On + \alpha)$? Natural candidates for $R(On + 1)$ seem deficient in that membership in the class may be fuzzy or else which sets belong to a particular member of a class in $R(On + 1)$ may be indeterminate. The real $R(On + 1)$ bears a relation to $R(On)$ somewhat analogous to that between the general recursive functions and the primitive recursive functions. The primitive recursive functions are generated by an inductive construction, whereas general recursive functions are difficult to extract from a larger collection, the partial recursive functions, which contains fuzzy objects. I suspect that one has to get by with fuzzy versions of $R(On + 1)$ and $R(On + \alpha)$. Nevertheless it is possible that some sense can be made of and some justification can be given for reflection principles such as the assertion that an α exists with $R(\alpha + 1)$ an elementary substructure of $R(On + 1)$.

Perhaps it would be helpful to summarize my position on sets and classes by recalling the points where my views agree and disagree with those put forward by Charles Parsons.

I think that Parsons has made a correct and important point in observing that the set theorist's use of proper classes -- even when he is working in a formal theory without proper classes -- is not at all as innocent as it appears.

I agree with Parsons that the notion of set is not dependent upon the notion of class and probably cannot be derived from it. I might state the difference very, very loosely by saying that "set" is a mathematical concept and "class" is a logical concept.

I take issue with Parson's attempt to assimilate the notion of class to that of set by pointing to an ambiguity in the notion of set.

In my view the most important aspect of the problem of proper classes is that we need to be able to talk about the truth of set theoretic propositions within our theory. The introduction of proper classes, if we can justify it, allows us to do this. But if we are to have a satisfactory theory of sets and classes, we should be able to talk of the truth of class theoretic propositions within our theory. Ultimately a satisfactory theory should have the property that truth for sentences of the theory is definable within the theory -- the theorem on the undefinability of truth notwithstanding. In order to do this, the most reasonable procedure seems to me to allow that some sentences of our theory have indeterminate truth values, in the manner I have sketched. Such a procedure is really a standard idea in logic. A related instance of it is the λ-calculus.